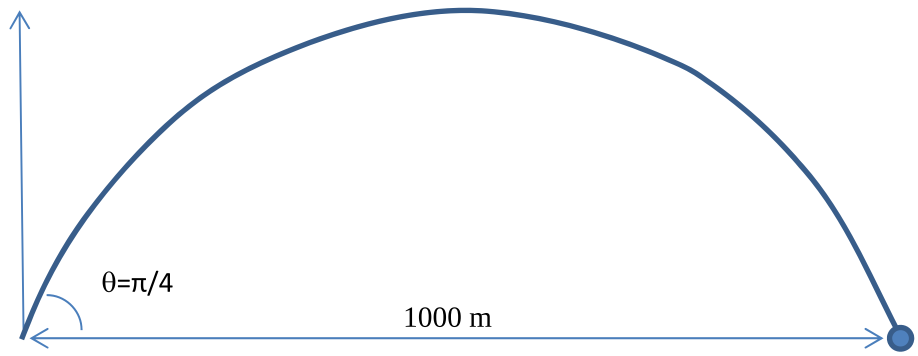
1. **(12 pts)** A projectile is launched from ground level at an angle θ=π/4, hitting a target 1000m away, also at ground level. What is the initial speed of the projectile? You may use an approximate value of the acceleration of free fall g=10 m/s^2.



$$\begin{cases} x = v_o \cos\theta\, t = \dfrac{v_o\, t}{\sqrt{2}} = 1000 & \Rightarrow\ t = \dfrac{1000\sqrt{2}}{v_o} \\ y = v_o \sin\theta\, t - \dfrac{g t^2}{2} = t\left(\dfrac{v_o}{\sqrt{2}} - \dfrac{g t}{2}\right) & \Rightarrow\ v_o = \dfrac{g t}{\sqrt{2}} = \dfrac{1000\, g}{v_o} \Rightarrow v_o^2 = 1000g \Rightarrow \boxed{v_o \approx 100\dfrac{m}{s}} \end{cases}$$

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2. **(12 pts)** Consider an object moving with acceleration $\mathbf{a}(t) = \left\langle \dfrac{1}{\sqrt{t+1}},\ e^{2t} \right\rangle$. Find its velocity and position if its initial velocity is $\mathbf{v}(0) = \left\langle 1,\ \dfrac{1}{2} \right\rangle$ and its initial position is $\mathbf{r}(0) = \left\langle \dfrac{1}{3},\ 1 \right\rangle$

$$\mathbf{a}(t) = \left\langle \frac{1}{\sqrt{t+1}},\ e^{2t} \right\rangle \Rightarrow \mathbf{v}(t) = \int \left\langle \frac{1}{\sqrt{t+1}},\ e^{2t} \right\rangle dt = \left\langle 2\sqrt{t+1}, \frac{e^{2t}}{2} \right\rangle + \mathbf{C}_1$$

$$\Rightarrow \mathbf{v}(0) = \left\langle 2,\ \frac{1}{2} \right\rangle + \mathbf{C}_1 = \left\langle 1,\ \frac{1}{2} \right\rangle \Rightarrow \mathbf{C}_1 = \langle -1,\ 0 \rangle \Rightarrow \boxed{\mathbf{v}(t) = \left\langle 2\sqrt{t+1} - 1,\ \frac{e^{2t}}{2} \right\rangle}$$

$$\Rightarrow \mathbf{r}(t) = \int \left\langle 2\sqrt{t+1} - 1, \frac{e^{2t}}{2} \right\rangle dt = \left\langle \frac{4}{3}(t+1)^{3/2} - t,\ \frac{e^{2t}}{4} \right\rangle + \mathbf{C}_2$$

$$\Rightarrow \mathbf{r}(0) = \left\langle \frac{4}{3},\ \frac{1}{4} \right\rangle + \mathbf{C}_2 = \left\langle \frac{1}{3},\ 1 \right\rangle \Rightarrow \mathbf{C}_2 = \left\langle \frac{1}{3},\ 1 \right\rangle - \left\langle \frac{4}{3},\ \frac{1}{4} \right\rangle = \left\langle -1,\ \frac{3}{4} \right\rangle \Rightarrow \boxed{\mathbf{r}(t) = \left\langle \frac{4}{3}(t+1)^{3/2} - t - 1,\ \frac{e^{2t} + 3}{4} \right\rangle}$$

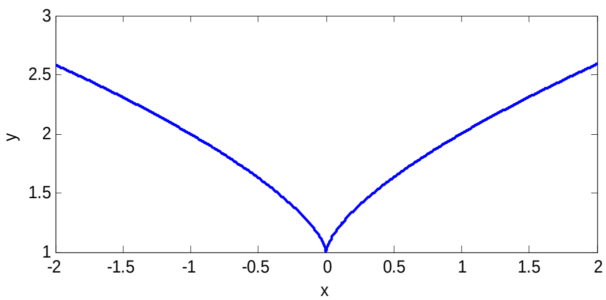
3. (12 pts) Calculate the length of this curve (hint: factor out *t* from the integrand, and make a simple substitution):

$$\mathbf{r}(t) = \langle t^3,\ 1+t^2 \rangle,\ t \in [0,\ 1]$$

$$\mathbf{v}(\mathrm{t}) = \langle 3t^2,\ 2t \rangle$$

$$\Rightarrow\ L = \int_0^1 |\mathbf{v}|\,dt = \int_0^1 \sqrt{9t^4+4t^2}\,dt = \int_0^1 \sqrt{9t^4+4t^2}\,dt = \int_0^1 t\sqrt{\underbrace{9t^2+4}_{u}}\,dt = \frac{1}{18}\int_4^{13}\sqrt{u}\,du = \frac{1}{27}u^{3/2}\Big|_4^{13} = \boxed{\frac{13^{3/2}-8}{27}}$$

Extra credit: solve for y: $y(x)=1+x^{2/3}$; this is a fractional power shifted up by 1; note the cusp at x=0, y=1:



4. (14 pts) Choose any two of the three limits: find the limit, or show that it does not exist

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ Let's check a couple different paths to the limit point:

$$\left.\begin{array}{l} x=0,\ \ y\to 0:\quad f(0,y)=\dfrac{0^2-y^2}{0^2+y^2}=-1 \\ y=0,\ \ x\to 0:\quad f(x,\ 0)=\dfrac{x^2-0^2}{x^2+0^2}=1 \end{array}\right\} \Rightarrow \boxed{\text{Does not exist}}$$

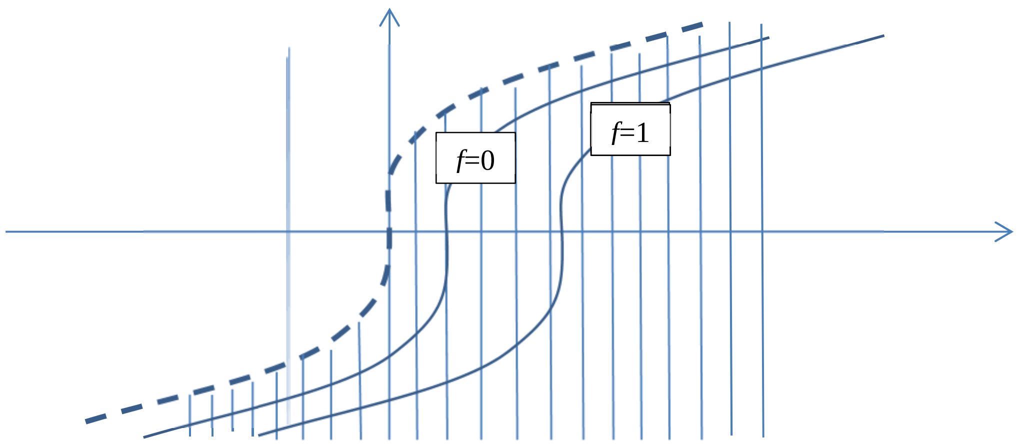
(b) $$\lim_{(x,y)\to(1,-1)} \frac{x^4-x^2y^2}{x^2+xy} = \lim_{(x,y)\to(1,-1)} \frac{x^2}{x}\,\frac{\overbrace{x^2-y^2}^{(x-y)(x+y)}}{x+y} = \lim_{(x,y)\to(1,-1)} x(x-y) = 1(1+1) = \boxed{2}$$

(c) $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{(xy)^2} = \lim_{z\to 0}\frac{1-\cos z}{z^2} \text{ Apply l'Hospital's rule: } = \lim_{z\to 0}\frac{\sin z}{2z} = \boxed{\frac{1}{2}}$$

5. (25 pts) For the function $f(x,y)=\ln\left(x-y^3\right)$

Domain: $x-y^3>0 \;\Rightarrow\; y<x^{1/3}$ (open set)

Range: $(-\infty,\ +\infty)$



Level curves: $\ln\left(x-y^3\right)=c \;\Rightarrow\; x-y^3=e^c \;\Rightarrow\; y=\left(x-e^c\right)^{1/3} \;\Rightarrow\; \begin{cases} f=0:\; y=\left(x-1\right)^{1/3} \\ f=1:\; y=\left(x-e\right)^{1/3} \end{cases}$

Find the linearization of the function at point (2, 1), and use it to estimate *f*(1.8, 1.1)

$$\begin{cases} f_x=\dfrac{1}{x-y^3} \\ f_y=\dfrac{-3y^2}{x-y^3} \end{cases} \Rightarrow \begin{cases} f(2,\,1)=\ln 1=0 \\ f_x(2,\,1)=1 \\ f_y(2,\,1)=-3 \end{cases}$$

$$\Rightarrow\; f(x,y)\approx f(x_o,y_o)+f(x_o,y_o)\left(x-x_o\right)+f(x_o,y_o)\left(y-y_o\right)\approx 0+\left(x-2\right)-3\left(y-1\right)$$

$$\Rightarrow\; f(1.8,1.1)\approx -0.2-3\left(0.1\right)=\boxed{-0.5}$$

a. Find the rate of change of *f*(*x*,*y*) at point (2, 1) in the direction of vector $-2\mathbf{i}+3\mathbf{j}$

$$\left.\begin{array}{l} \nabla f(1,\,1)=\left\langle 1,\ -3\right\rangle \\ \hat{v}=\dfrac{\mathbf{v}}{|\,\mathbf{v}\,|}=\dfrac{\left\langle -2,\,3\right\rangle}{\sqrt{13}} \end{array}\right\} \quad D_v f(2,\,1)=\frac{-2-9}{\sqrt{13}}=\boxed{-\frac{11}{\sqrt{13}}}$$

b. Find the unit vector in the direction of zero change of *f*(*x*,*y*) at point (2, 1)

It is clear that $\mathbf{u}=\left\langle 3,\,1\right\rangle$ would give zero dot product with vector $\nabla f(2,\,1)=\left\langle 1,\ -3\right\rangle$

Thus, the answer is $\boxed{\hat{\mathbf{u}}=\dfrac{\left\langle 3,\,1\right\rangle}{\sqrt{10}}=\left\langle \dfrac{3}{\sqrt{10}},\,\dfrac{1}{\sqrt{10}}\right\rangle}$

6. **(12 pts)** Use the chain rule to evaluate $\frac{\partial f}{\partial t}$ at $s=t=1$

$$x = s/t,\ \ y = s-t \ \Rightarrow\ x(1,1)=1,\ y(1,1)=0,\ \begin{cases} \dfrac{\partial x}{\partial t} = -\left.\dfrac{s}{t^2}\right|_{(1,\,1)} = -1 \\ \dfrac{\partial y}{\partial t} = -1 \end{cases}$$

$$f = 3x + y + e^{\sin\left(y^2\right)/x} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 3 - \dfrac{\sin\left(y^2\right)}{x^2} e^{\sin\left(y^2\right)/x} \\ \dfrac{\partial f}{\partial y} = 1 + \dfrac{2y\,\cos\left(y^2\right)}{x} e^{\sin\left(y^2\right)/x} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}(1,0) = 3 \\ \dfrac{\partial f}{\partial y}(1,0) = 1 \end{cases}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 3\cdot(-1) - 1 = \boxed{-4}$$

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7. **(12 pts)** Find the local maxima, minima and saddle points of

$$f(x,y) = xy + 2x - \ln\left(x^2 y\right) = xy + 2x - 2\ln x - \ln y$$

$$\begin{cases} f_x = y + 2 - \dfrac{2}{x} = 0 \\ f_y = x - \dfrac{1}{y} = 0 \ \Rightarrow\ x = \dfrac{1}{y} \end{cases} \Rightarrow \begin{cases} y + 2 - 2y = 2 - y = 0 \ \Rightarrow\ \boxed{y=2} \\ \boxed{x = \dfrac{1}{y} = \dfrac{1}{2}} \end{cases}$$

Note: although derivatives don't exist at $x = 0$ or $y = 0$, these coordinates are outside of the domain

Thus, there is only one cirtical point to check: $\left(\frac{1}{2},\ 2\right)$

$$\Rightarrow \begin{cases} f_{xx} = \dfrac{2}{x^2} \\ f_{yy} = \dfrac{1}{y^2} \\ f_{xy} = 1 \end{cases} \Rightarrow \begin{cases} f_{xx}\left(\dfrac{1}{2},\ 2\right) = 8 > 0 \\ f_{yy}\left(\dfrac{1}{2},\ 2\right) = \dfrac{1}{4} \\ f_{xy} = 1 \end{cases} \Rightarrow\ H = \begin{vmatrix} 8 & 1 \\ 1 & 1/4 \end{vmatrix} = 2 - 1 = 1 > 0 \ \Rightarrow\ \boxed{\text{Local min}}$$